# 6 GRAVITATION AND UNIFORM CIRCULAR MOTION


**Figure 6.1** This Australian Grand Prix Formula 1 race car moves in a circular path as it makes the turn. Its wheels also spin rapidly—the latter completing many revolutions, the former only part of one (a circular arc). The same physical principles are involved in each. (credit: Richard Munckton)

## Chapter Outline



## Connection for AP® Courses

Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. This chapter supports Big Idea 3 that interactions between objects are described by forces, and thus change in motion is a result of a net force exerted on an object. In this chapter, this idea is applied to uniform circular motion. In some ways, this chapter is a continuation of **Dynamics: Newton's Laws of Motion** as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, which is motion in a circular path at constant speed. As an object moves on a circular path, the magnitude of its velocity remains constant, but the direction of the velocity is changing. This means there is an acceleration that we will refer to as a “centripetal” acceleration caused by a net external force, also called the “centripetal” force (Enduring Understanding 3.B). The centripetal force is the net force totaling all external forces acting on the object (Essential Knowledge 3.B.1). In order to determine the net force, a free-body diagram may be useful (Essential Knowledge 3.B.2).

Studying this topic illustrates most of the concepts associated with rotational motion and leads to many new topics we group under the name *rotation*. This motion can be described using kinematics variables (Essential Knowledge 3.A.1), but in addition to linear variables, we will introduce angular variables. We use various ways to describe motion, namely, verbally, algebraically and graphically (Learning Objective 3.A.1.1). Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving over ice. Some combinations of both types of motion are conveniently described with fictitious forces which appear as a result of using a non-inertial frame of reference (Enduring Understanding 3.A).

Furthermore, the properties of uniform circular motion can be applied to the motion of massive objects in a gravitational field. Thus, this chapter supports Big Idea 1 that gravitational mass is an important property of an object or a system.

We have experimental evidence that gravitational and inertial masses are equal (Enduring Understanding 1.C), and that gravitational mass is a measure of the strength of the gravitational interaction (Essential Knowledge 1.C.2). Therefore, this chapter will support Big Idea 2 that fields existing in space can be used to explain interactions, because any massive object creates a gravitational field in space (Enduring Understanding 2.B). Mathematically, we use Newton's universal law of gravitation to provide a model for the gravitational interaction between two massive objects (Essential Knowledge 2.B.2). We will discover that this model describes the interaction of one object with mass with another object with mass (Essential Knowledge 3.C.1), and also that gravitational force is a long-range force (Enduring Understanding 3.C).

The concepts in this chapter support:

**Big Idea 1** Objects and systems have properties such as mass and charge. Systems may have internal structure.

Enduring Understanding 1.C Objects and systems have properties of inertial mass and gravitational mass that are experimentally verified to be the same and that satisfy conservation principles.

Essential Knowledge 1.C.2 Gravitational mass is the property of an object or a system that determines the strength of the gravitational interaction with other objects, systems, or gravitational fields.

Essential Knowledge 1.C.3 Objects and systems have properties of inertial mass and gravitational mass that are experimentally verified to be the same and that satisfy conservation principles.

**Big Idea 2** Fields existing in space can be used to explain interactions.

Enduring Understanding 2.B A gravitational field is caused by an object with mass.

Essential Knowledge 2.B.2. The gravitational field caused by a spherically symmetric object with mass is radial and, outside the object, varies as the inverse square of the radial distance from the center of that object.

**Big Idea 3** The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.1. An observer in a particular reference frame can describe the motion of an object using such quantities as position, displacement, distance, velocity, speed, and acceleration.

Essential Knowledge 3.A.3. A force exerted on an object is always due to the interaction of that object with another object.

Enduring Understanding 3.B Classically, the acceleration of an object interacting with other objects can be predicted by using $a = \sum F / m$.

Essential Knowledge 3.B.1 If an object of interest interacts with several other objects, the net force is the vector sum of the individual forces.

Essential Knowledge 3.B.2 Free-body diagrams are useful tools for visualizing forces being exerted on a single object and writing the equations that represent a physical situation.

Enduring Understanding 3.C At the macroscopic level, forces can be categorized as either long-range (action-at-a-distance) forces or contact forces.

Essential Knowledge 3.C.1. Gravitational force describes the interaction of one object that has mass with another object that has mass.

## 6.1 Rotation Angle and Angular Velocity

### Learning Objectives

By the end of this section, you will be able to:

- Define arc length, rotation angle, radius of curvature, and angular velocity.
- Calculate the angular velocity of a car wheel spin.

In **Kinematics**, we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration.

**Two-Dimensional Kinematics** dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

## Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in **Figure 6.2** rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r}. \tag{6.1}$$


**Figure 6.2** All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta\theta$ in a time $\Delta t$ .

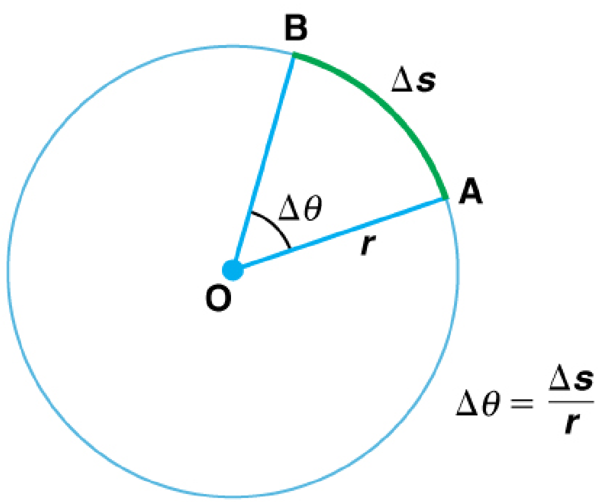


**Figure 6.3** The radius of a circle is rotated through an angle $\Delta\theta$ . The arc length $\Delta s$ is described on the circumference.

The **arc length** $\Delta s$ is the distance traveled along a circular path as shown in **Figure 6.3** Note that $r$ is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius $r$ . The circumference of a circle is $2\pi r$ . Thus for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi. \tag{6.2}$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

$$2\pi \text{ rad} = 1 \text{ revolution}. \tag{6.3}$$

A comparison of some useful angles expressed in both degrees and radians is shown in **Table 6.1**.

Table 6.1 Comparison of Angular Units

| Degree Measures | Radian Measure |
|---|---|
| $30^\circ$ | $\frac{\pi}{6}$ |
| $60^\circ$ | $\frac{\pi}{3}$ |
| $90^\circ$ | $\frac{\pi}{2}$ |
| $120^\circ$ | $\frac{2\pi}{3}$ |
| $135^\circ$ | $\frac{3\pi}{4}$ |
| $180^\circ$ | $\pi$ |

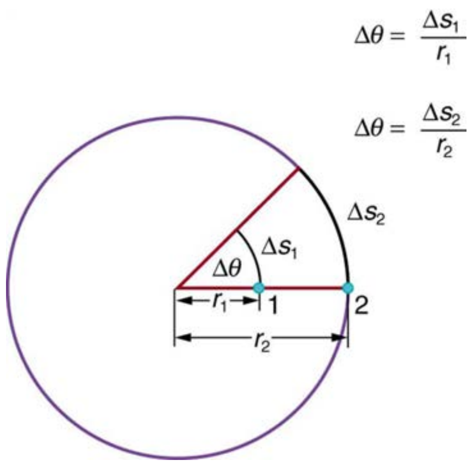


**Figure 6.4** Points 1 and 2 rotate through the same angle ( $\Delta\theta$ ), but point 2 moves through a greater arc length $(\Delta s)$ because it is at a greater distance from the center of rotation $(r)$ .

If $\Delta\theta = 2\pi$ rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are $360^\circ$ in a circle or one revolution, the relationship between radians and degrees is thus

$$2\pi \text{ rad} = 360^\circ \tag{6.4}$$

so that

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ. \tag{6.5}$$

## Angular Velocity

How fast is an object rotating? We define **angular velocity** $\omega$ as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t}, \tag{6.6}$$

where an angular rotation $\Delta\theta$ takes place in a time $\Delta t$ . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity $\omega$ is analogous to linear velocity $v$ . To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length $\Delta s$ in a time $\Delta t$ , and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}. \tag{6.7}$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$ . Substituting this into the expression for $v$ gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega. \tag{6.8}$$

We write this relationship in two different ways and gain two different insights:

$$v = r\omega \text{ or } \omega = \frac{v}{r}. \tag{6.9}$$

The first relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ states that the linear velocity $v$ is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest $r$), as you might expect. We can also call this linear speed $v$ of a point on the rim the *tangential speed*. The second relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed $v$ of the car. See **Figure 6.5**. So the faster the car moves, the faster the tire spins—large $v$ means a large $\omega$, because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity ($\omega$) will produce a greater linear speed ($v$) for the car.

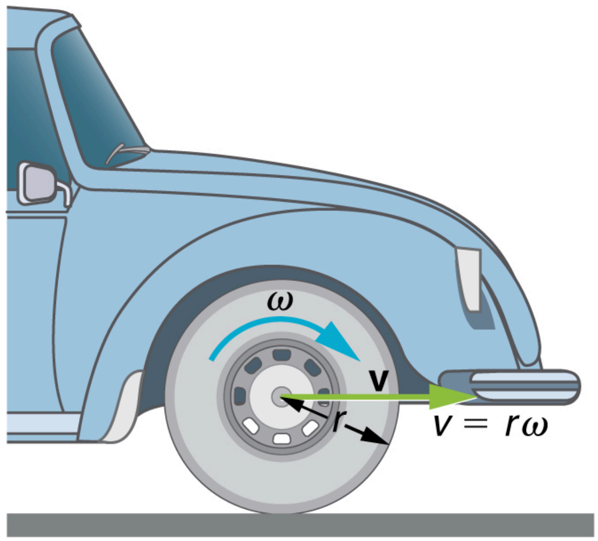


**Figure 6.5** A car moving at a velocity $v$ to the right has a tire rotating with an angular velocity $\omega$ .The speed of the tread of the tire relative to the axle is $v$, the same as if the car were jacked up. Thus the car moves forward at linear velocity $v = r\omega$, where $r$ is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

## Example 6.1 How Fast Does a Car Tire Spin?

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at $15.0\text{ m/s}$ (about $54\text{ km/h}$). See **Figure 6.5**.

**Strategy**

Because the linear speed of the tire rim is the same as the speed of the car, we have $v = 15.0\text{ m/s}.$ The radius of the tire is given to be $r = 0.300\text{ m}.$ Knowing $v$ and $r$, we can use the second relationship in $v = r\omega, \omega = \frac{v}{r}$ to calculate the angular velocity.

**Solution**

To calculate the angular velocity, we will use the following relationship:

$$\omega = \frac{v}{r}. \tag{6.10}$$

Substituting the knowns,

$$\omega = \frac{15.0\text{ m/s}}{0.300\text{ m}} = 50.0\text{ rad/s}. \tag{6.11}$$

**Discussion**

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

$$\omega = (15.0\text{ m/s}) / (1.20\text{ m}) = 12.5\text{ rad/s}. \tag{6.12}$$

Both $\omega$ and $v$ have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in **Figure 6.6**.

Take-Home Experiment

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.

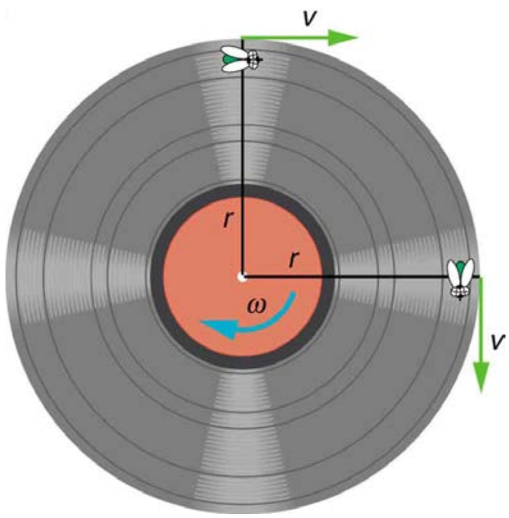


**Figure 6.6** As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.

PhET Explorations: Ladybug Revolution

PhET Interactive Simulation

**Figure 6.7 Ladybug Revolution (http://cnx.org/content/m54992/1.2/rotation_en.jar)**

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

## 6.2 Centripetal Acceleration

### Learning Objectives

By the end of this section, you will be able to:

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

**Figure 6.8** shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration**( $a_c$ ); centripetal means “toward the center” or “center seeking.”