CHAPTER 6
Circular and Rotational Motion

Figure 6.1 This Australian Grand Prix Formula 1 race car moves in a circular path as it makes the turn. Its wheels also spin rapidly. The same physical principles are involved in both of these motions. (Richard Munckton).

Chapter Outline

INTRODUCTION You may recall learning about various aspects of motion along a straight line: kinematics (where we learned about displacement, velocity, and acceleration), projectile motion (a special case of two-dimensional kinematics), force, and Newton's laws of motion. In some ways, this chapter is a continuation of Newton's laws of motion. Recall that Newton's first law tells us that objects move along a straight line at constant speed unless a net external force acts on them. Therefore, if an object moves along a circular path, such as the car in the photo, it must be experiencing an external force. In this chapter, we explore both circular motion and rotational motion.

6.1 Angle of Rotation and Angular Velocity

Section Learning Objectives

By the end of this section, you will be able to do the following:

- Describe the angle of rotation and relate it to its linear counterpart
- Describe angular velocity and relate it to its linear counterpart
- Solve problems involving angle of rotation and angular velocity

Section Key Terms

angle of rotation	angular velocity	arc length	circular motion
radius of curvature	rotational motion	spin	tangential velocity

Angle of Rotation

What exactly do we mean by *circular motion* or *rotation*? **Rotational motion** is the circular motion of an object about an axis of rotation. We will discuss specifically circular motion and spin. **Circular motion** is when an object moves in a circular path. Examples of circular motion include a race car speeding around a circular curve, a toy attached to a string swinging in a circle around your head, or the circular *loop-the-loop* on a roller coaster. **Spin** is rotation about an axis that goes through the center of mass of the object, such as Earth rotating on its axis, a wheel turning on its axle, the spin of a tornado on its path of destruction, or a figure skater spinning during a performance at the Olympics. Sometimes, objects will be spinning while in circular motion, like the Earth spinning on its axis while revolving around the Sun, but we will focus on these two motions separately.

When solving problems involving rotational motion, we use variables that are similar to linear variables (distance, velocity, acceleration, and force) but take into account the curvature or rotation of the motion. Here, we define the **angle of rotation**, which is the angular equivalence of distance; and **angular velocity**, which is the angular equivalence of linear velocity.

When objects rotate about some axis—for example, when the CD in Figure 6.2 rotates about its center—each point in the object follows a circular path.

Figure 6.2 All points on a CD travel in circular paths. The pits (dots) along a line from the center to the edge all move through the same angle $\Delta\theta$ in time Δt .

The **arc length,** , is the distance traveled along a circular path. The **radius of curvature**, **r**, is the radius of the circular path. Both are shown in Figure 6.3.

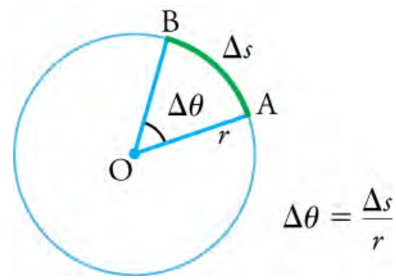

Figure 6.3 The radius (r) of a circle is rotated through an angle $\Delta\theta$. The arc length, Δs, is the distance covered along the circumference.

Consider a line from the center of the CD to its edge. In a given time, each *pit* (used to record information) on this line moves through the same angle. The angle of rotation is the amount of rotation and is the angular analog of distance. The angle of rotation $\Delta\theta$ is the arc length divided by the radius of curvature.

$$\Delta\theta = \frac{\Delta s}{r}$$

The angle of rotation is often measured by using a unit called the radian. (Radians are actually dimensionless, because a radian is defined as the ratio of two distances, radius and arc length.) A revolution is one complete rotation, where every point on the circle returns to its original position. One revolution covers 2π radians (or 360 degrees), and therefore has an angle of rotation of 2π radians, and an arc length that is the same as the circumference of the circle. We can convert between radians, revolutions, and degrees using the relationship

1 revolution = 2π rad = 360°. See Table 6.1 for the conversion of degrees to radians for some common angles.

$$\begin{aligned} 2\pi \text{ rad} &= 360^\circ \\ 1 \text{ rad} &= \frac{360^\circ}{2\ \pi} \approx 57.3^\circ \end{aligned} \tag{6.1}$$

Degree Measures	Radian Measures
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	π

Table 6.1 Commonly Used Angles in Terms of Degrees and Radians

Angular Velocity

How fast is an object rotating? We can answer this question by using the concept of angular velocity. Consider first the angular speed (ω) is the rate at which the angle of rotation changes. In equation form, the angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t}, \tag{6.2}$$

which means that an angular rotation ($\Delta\theta$) occurs in a time, Δt. If an object rotates through a greater angle of rotation in a given time, it has a greater angular speed. The units for angular speed are radians per second (rad/s).

Now let's consider the direction of the angular speed, which means we now must call it the angular velocity. The direction of the angular velocity is along the axis of rotation. For an object rotating clockwise, the angular velocity points away from you along the axis of rotation. For an object rotating counterclockwise, the angular velocity points toward you along the axis of rotation.

Angular velocity (ω) is the angular version of linear velocity **v**. **Tangential velocity** is the instantaneous linear velocity of an object in rotational motion**.** To get the precise relationship between angular velocity and tangential velocity, consider again a pit on the rotating CD. This pit moves through an arc length (Δs) in a *short* time (Δt) so its tangential *speed* is

$$v = \frac{\Delta s}{\Delta t}. \tag{6.3}$$

From the definition of the angle of rotation, $\Delta\theta = \frac{\Delta s}{r}$, we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for *v* gives

$$v = \frac{\mathrm{r}\Delta\theta}{\Delta t} = r\omega.$$

The equation $v = r\omega$ says that the tangential speed *v* is proportional to the distance *r* from the center of rotation. Consequently, tangential speed is greater for a point on the outer edge of the CD (with larger *r*) than for a point closer to the center of the CD (with smaller *r*). This makes sense because a point farther out from the center has to cover a longer arc length in the same amount of time as a point closer to the center. Note that both points will still have the same angular speed, regardless of their distance from the center of rotation. See Figure 6.4.

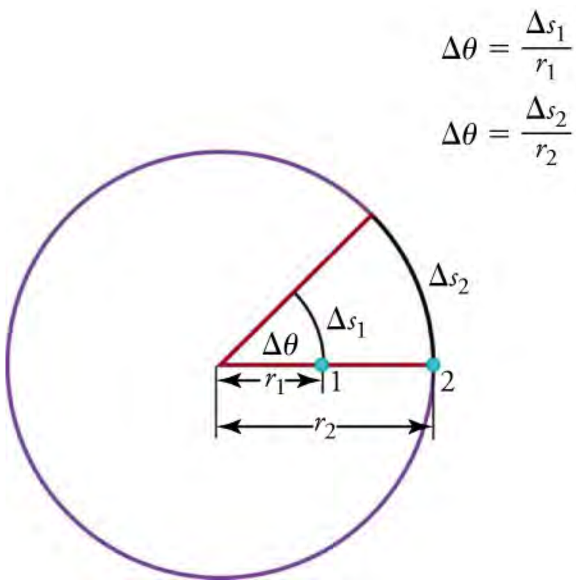

Figure 6.4 Points 1 and 2 rotate through the same angle ($\Delta\theta$), but point 2 moves through a greater arc length (Δs_2) because it is farther from the center of rotation.

Now, consider another example: the tire of a moving car (see Figure 6.5). The faster the tire spins, the faster the car moves—large ω means large *v* because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity, **ω**, will produce a greater linear (tangential) velocity, **v,** for the car. This is because a larger radius means a longer arc length must contact the road, so the car must move farther in the same amount of time.

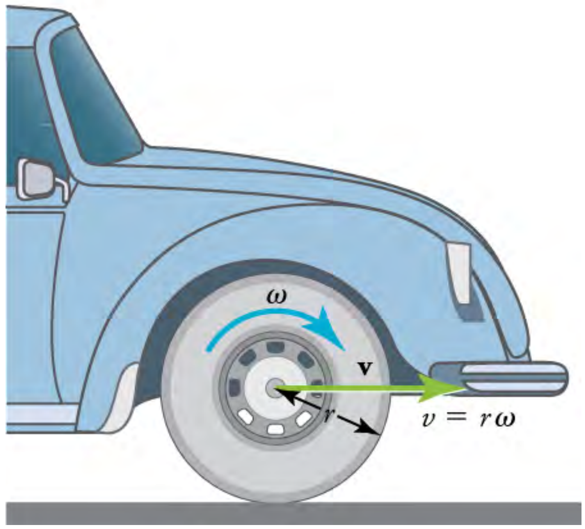

Figure 6.5 A car moving at a velocity, **v,** to the right has a tire rotating with angular velocity $\boldsymbol{\omega}$. The speed of the tread of the tire relative to the axle is *v*, the same as if the car were jacked up and the wheels spinning without touching the road. Directly below the axle, where the tire touches the road, the tire tread moves backward with respect to the axle with tangential velocity $v = r\omega$, where *r* is the tire radius. Because the road is stationary with respect to this point of the tire, the car must move forward at the linear velocity **v**. A larger angular velocity for the tire means a greater linear velocity for the car.

However, there are cases where linear velocity and tangential velocity are not equivalent, such as a car spinning its tires on ice. In this case, the linear velocity will be less than the tangential velocity. Due to the lack of friction under the tires of a car on ice, the arc length through which the tire treads move is greater than the linear distance through which the car moves. It's similar to running on a treadmill or pedaling a stationary bike; you are literally going nowhere fast.

TIPS FOR SUCCESS

Angular velocity $\boldsymbol{\omega}$ and tangential velocity **v** are vectors, so we must include magnitude and direction. The direction of the angular velocity is along the axis of rotation, and points away from you for an object rotating clockwise, and toward you for an object rotating counterclockwise. In mathematics this is described by the right-hand rule. Tangential velocity is usually described as up, down, left, right, north, south, east, or west, as shown in Figure 6.6.

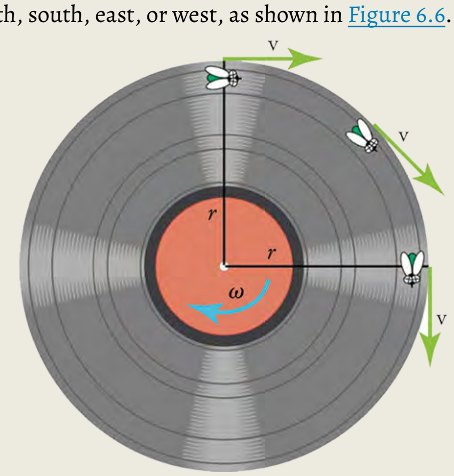

Figure 6.6 As the fly on the edge of an old-fashioned vinyl record moves in a circle, its instantaneous velocity is always at a tangent to the circle. The direction of the angular velocity is into the page this case.

WATCH PHYSICS

Relationship between Angular Velocity and Speed

This video reviews the definition and units of angular velocity and relates it to linear speed. It also shows how to convert between revolutions and radians.

Click to view content (https://www.youtube.com/embed/zAx61CO5mDw)

GRASP CHECK

For an object traveling in a circular path at a constant angular speed, would the linear speed of the object change if the radius of the path increases?

a. Yes, because tangential speed is independent of the radius.
b. Yes, because tangential speed depends on the radius.
c. No, because tangential speed is independent of the radius.
d. No, because tangential speed depends on the radius.

Solving Problems Involving Angle of Rotation and Angular Velocity

Snap Lab

Measuring Angular Speed

In this activity, you will create and measure uniform circular motion and then contrast it with circular motions with different radii.

- One string (1 m long)
- One object (two-hole rubber stopper) to tie to the end
- One timer

Procedure

1. Tie an object to the end of a string.
2. Swing the object around in a horizontal circle above your head (swing from your wrist). It is important that the circle be horizontal!
3. Maintain the object at uniform speed as it swings.
4. Measure the angular speed of the object in this manner. Measure the time it takes in seconds for the object to travel 10 revolutions. Divide that time by 10 to get the angular speed in revolutions per second, which you can convert to radians per second.
5. What is the approximate linear speed of the object?
6. Move your hand up the string so that the length of the string is 90 cm. Repeat steps 2–5.
7. Move your hand up the string so that its length is 80 cm. Repeat steps 2–5.
8. Move your hand up the string so that its length is 70 cm. Repeat steps 2–5.
9. Move your hand up the string so that its length is 60 cm. Repeat steps 2–5
10. Move your hand up the string so that its length is 50 cm. Repeat steps 2–5
11. Make graphs of angular speed vs. radius (i.e. string length) and linear speed vs. radius. Describe what each graph looks like.

GRASP CHECK

If you swing an object slowly, it may rotate at less than one revolution per second. What would be the revolutions per second for an object that makes one revolution in five seconds? What would be its angular speed in radians per second?

a. The object would spin at $\frac{1}{5}$ rev/s. The angular speed of the object would be $\frac{2\pi}{5}$ rad/s.
b. The object would spin at $\frac{1}{5}$ rev/s. The angular speed of the object would be $\frac{\pi}{5}$ rad/s.

c. The object would spin at 5 rev/s. The angular speed of the object would be $10\pi \text{ rad/s}$.
d. The object would spin at 5 rev/s. The angular speed of the object would be $5\pi \text{ rad/s}$.

Now that we have an understanding of the concepts of angle of rotation and angular velocity, we'll apply them to the real-world situations of a clock tower and a spinning tire.

WORKED EXAMPLE

Angle of rotation at a Clock Tower

The clock on a clock tower has a radius of 1.0 m. (a) What angle of rotation does the hour hand of the clock travel through when it moves from 12 p.m. to 3 p.m.? (b) What's the arc length along the outermost edge of the clock between the hour hand at these two times?

Strategy

We can figure out the angle of rotation by multiplying a full revolution (2π radians) by the fraction of the 12 hours covered by the hour hand in going from 12 to 3. Once we have the angle of rotation, we can solve for the arc length by rearranging the equation $\Delta\theta = \frac{\Delta s}{r}$ since the radius is given.

Solution to (a)

In going from 12 to 3, the hour hand covers 1/4 of the 12 hours needed to make a complete revolution. Therefore, the angle between the hour hand at 12 and at 3 is $\frac{1}{4} \times 2 \ \pi\text{rad} = \frac{\pi}{2}$ (i.e., 90 degrees).

Solution to (b)

Rearranging the equation

$$\Delta\theta = \frac{\Delta s}{r}, \tag{6.4}$$

we get

$$\Delta s = r\Delta\theta. \tag{6.5}$$

Inserting the known values gives an arc length of

$$\begin{aligned} \Delta s &= (1.0 \text{ m})\left(\tfrac{\pi}{2}\text{rad}\right) \\ &= 1.6 \text{ m} \end{aligned} \tag{6.6}$$

Discussion

We were able to drop the radians from the final solution to part (b) because radians are actually dimensionless. This is because the radian is defined as the ratio of two distances (radius and arc length). Thus, the formula gives an answer in units of meters, as expected for an arc length.

WORKED EXAMPLE

How Fast Does a Car Tire Spin?

Calculate the angular speed of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See Figure 6.5.

Strategy

In this case, the speed of the tire tread with respect to the tire axle is the same as the speed of the car with respect to the road, so we have v = 15.0 m/s. The radius of the tire is r = 0.300 m. Since we know v and r, we can rearrange the equation $v = r\omega$, to get $\omega = \frac{v}{r}$ and find the angular speed.

Solution

To find the angular speed, we use the relationship: $\omega = \frac{v}{r}$.

Inserting the known quantities gives

$$\begin{aligned}\omega &= \frac{15.0\ \text{m/s}}{0.300\ \text{m}} \\ &= 50.0\ \text{rad/s}.\end{aligned} \tag{6.7}$$

Discussion

When we cancel units in the above calculation, we get 50.0/s (i.e., 50.0 per second, which is usually written as 50.0 s^{-1}). But the angular speed must have units of rad/s. Because radians are dimensionless, we can insert them into the answer for the angular speed because we know that the motion is circular. Also note that, if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular speed of

$$\begin{aligned}\omega &= \frac{15.0\ \text{m/s}}{1.20\ \text{m}} \\ &= 12.5\ \text{rad/s}\end{aligned} \tag{6.8}$$

Practice Problems

1. What is the angle in degrees between the hour hand and the minute hand of a clock showing 9:00 a.m.?
   a. 0°
   b. 90°
   c. 180°
   d. 360°

2. What is the approximate value of the arc length between the hour hand and the minute hand of a clock showing 10:00 a.m if the radius of the clock is 0.2 m?
   a. 0.1 m
   b. 0.2 m
   c. 0.3 m
   d. 0.6 m

Check Your Understanding

3. What is circular motion?
   a. Circular motion is the motion of an object when it follows a linear path.
   b. Circular motion is the motion of an object when it follows a zigzag path.
   c. Circular motion is the motion of an object when it follows a circular path.
   d. Circular motion is the movement of an object along the circumference of a circle or rotation along a circular path.

4. What is meant by radius of curvature when describing rotational motion?
   a. The radius of curvature is the radius of a circular path.
   b. The radius of curvature is the diameter of a circular path.
   c. The radius of curvature is the circumference of a circular path.
   d. The radius of curvature is the area of a circular path.

5. What is angular velocity?
   a. Angular velocity is the rate of change of the diameter of the circular path.
   b. Angular velocity is the rate of change of the angle subtended by the circular path.
   c. Angular velocity is the rate of change of the area of the circular path.
   d. Angular velocity is the rate of change of the radius of the circular path.

6. What equation defines angular velocity, ω? Take that r is the radius of curvature, θ is the angle, and t is time.
   a. $\omega = \frac{\Delta\theta}{\Delta t}$
   b. $\omega = \frac{\Delta t}{\Delta\theta}$
   c. $\omega = \frac{\Delta r}{\Delta t}$
   d. $\omega = \frac{\Delta t}{\Delta r}$

7. Identify three examples of an object in circular motion.

a. an artificial satellite orbiting the Earth, a race car moving in the circular race track, and a top spinning on its axis
b. an artificial satellite orbiting the Earth, a race car moving in the circular race track, and a ball tied to a string being swung in a circle around a person's head
c. Earth spinning on its own axis, a race car moving in the circular race track, and a ball tied to a string being swung in a circle around a person's head
d. Earth spinning on its own axis, blades of a working ceiling fan, and a top spinning on its own axis

8. What is the relative orientation of the radius and tangential velocity vectors of an object in uniform circular motion?
a. Tangential velocity vector is always parallel to the radius of the circular path along which the object moves.
b. Tangential velocity vector is always perpendicular to the radius of the circular path along which the object moves.
c. Tangential velocity vector is always at an acute angle to the radius of the circular path along which the object moves.
d. Tangential velocity vector is always at an obtuse angle to the radius of the circular path along which the object moves.

6.2 Uniform Circular Motion

Section Learning Objectives

By the end of this section, you will be able to do the following:

- Describe centripetal acceleration and relate it to linear acceleration
- Describe centripetal force and relate it to linear force
- Solve problems involving centripetal acceleration and centripetal force

Section Key Terms

centrifugal force	centripetal acceleration	centripetal force	uniform circular motion

Centripetal Acceleration

In the previous section, we defined circular motion. The simplest case of circular motion is **uniform circular motion**, where an object travels a circular path *at a constant speed*. Note that, unlike speed, the linear velocity of an object in circular motion is constantly changing because it is always changing direction. We know from kinematics that acceleration is a change in velocity, either in magnitude or in direction or both. Therefore, an object undergoing uniform circular motion is always accelerating, even though the magnitude of its velocity is constant.

You experience this acceleration yourself every time you ride in a car while it turns a corner. If you hold the steering wheel steady during the turn and move at a constant speed, you are executing uniform circular motion. What you notice is a feeling of sliding (or being flung, depending on the speed) away from the center of the turn. This isn't an actual force that is acting on you—it only happens because your body wants to continue moving in a straight line (as per Newton's first law) whereas the car is turning off this straight-line path. Inside the car it appears as if you are forced away from the center of the turn. This fictitious force is known as the **centrifugal force**. The sharper the curve and the greater your speed, the more noticeable this effect becomes.

Figure 6.7 shows an object moving in a circular path at constant speed. The direction of the instantaneous tangential velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity; in this case it points roughly toward the center of rotation. (The center of rotation is at the center of the circular path). If we imagine Δs becoming smaller and smaller, then the acceleration would point *exactly* toward the center of rotation, but this case is hard to draw. We call the acceleration of an object moving in uniform circular motion the **centripetal acceleration** $\mathbf{a}_c$ because centripetal means *center seeking*.